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Nonlinear Evolution Equations and *H* Theorems

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We prove a structure theorem for evolution equations in the state space of a discrete classical system fulfilling a class of H theorems. These H theorems are proved to give strong implications on the time behavior of the solutions. All the results are demonstrated by examples (Boltzmann-like equations, for example).

KEY WORDS: Dissipative processes; stochastic dynamics; evolution equations; H theorems.

1. INTRODUCTION

The dissipative time development of a physical system is often indicated by the monotone behavior of certain state functionals. The most famous example is the Boltzmann equation with Boltzmann's H theorem.[6]

In the following we consider classical systems with finitely many (pure) states i ($i = 1, \ldots, n$). A (mixed) *state* is an n-dimensional probability vector $\mathbf{p}$, i.e., $\mathbf{p} = (p_1, \ldots, p_n)$ with $p_i \geqslant 0$ for all i and $\sum_{i=1}^{n} p_i = 1$.

The set of all probability vectors (or states) will be called the *state space* P_n. The interior of the state space is formed by the *strictly positive states* (i.e., all components of such a state $\mathbf{p}$ are positive—briefly, $\mathbf{p} > \mathbf{0}$). All remaining states belong to the boundary.

We describe the time development by a *trajectory* in the state space P_n. This means that we have a map, which determines a state $\mathbf{p}(t)$ for every instant of time $t \geqslant 0$ [$\mathbf{p}(0)$ will be called the initial state]. As a special case we have that the time development is given by a system of ordinary differential equations. Differential equations will be called *evolution equations in the state space* P_n *or* P_n*-invariant evolution equations*, when every

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solution which starts with initial value from the state space P_n can be extended to all times $t > 0$ and never leave the state space P_n.

Important examples of evolution equations in the state space are Pauli's master equation[14] and the Boltzmann-like equations (B *equations*). The latter are equations of the following structure:

$$(d/dt)p_i = \sum_{j,k,l=1}^{n} (A_{ijkl}p_kp_l - A_{klij}p_ip_j), \qquad i = 1, \ldots, n \tag{1}$$

with the conditions for the scheme of the A_{ijkl}'s

$$A_{ijkl} = A_{jikl} = A_{ijlk} \geqslant 0, \qquad \text{for all } i, j, k, l \tag{2}$$

$$\sum_{i,j=1}^{n} A_{ijkl} = 1, \qquad \text{for all } k, l \tag{3}$$

Every solution of a B equation with initial value $\mathbf{p}(0) \in P_n$ can be extended to a global solution defined for all times t ($0 \leqslant t < \infty$) and never leave the state space P_n.[1]

Already Boltzmann[7] studied equations of such a type as discrete version of his spatially homogeneous equation. The coarse-grained description with the p_i's and A_{ijkl}'s originates, for instance, from integration of the distribution function and the interaction kernel, respectively, over the corresponding cells in the velocity space. We interpret A_{ijkl} as transition probability per unit time of a pair of particles from cells k, l to be scattered into cells i, j.

Then, following Boltzmann,[7] the discrete version of the H functional is

$$H(\mathbf{p}) = \sum_{i=1}^{n} p_i \log p_i$$

Let us now consider the structure of $H(\mathbf{p})$. One notices that

$$H(\mathbf{p}) = \sum_{i=1}^{n} (1/n) g(p_i/(1/n)) - \log n$$

when $g(s)$ represents the convex function $s \log s$. This functional will be generalized in various aspects:

Definition 1.1 (relative H functionals). Let g be an arbitrary convex function (defined on R_+) and $\mathbf{p}, \mathbf{q}$ be states. Then we define

$$S_g(\mathbf{p}; \mathbf{q}) := \sum_{i=1}^{n} q_i g(p_i/q_i)$$

Remark. By specializing the reference state as $\mathbf{q} = (1/n, \ldots, 1/n)$ and in choosing $g(s) = s \log s$, we arrive at H again (up to a constant).

Now we formulate relative H theorems.

Definition 1.2 (relative H theorems). We say that the $\mathbf{q}$-*relative H theorems are fulfilled* for a trajectory $(\mathbf{p}(t))_{t \geqslant 0}$ in the state space P_n if and only if for any convex function g

$$S_g(\mathbf{p}(t); \mathbf{q}) \leqslant S_g(\mathbf{p}(t'); \mathbf{q}) \qquad \text{when} \quad t \geqslant t', \quad \mathbf{q} \in P_n$$

With respect to this definition we may now raise the following question: Under what conditions are the $\mathbf{q}$-relative H theorems fulfilled for *every* state solution of a B equation? Examples and conditions where this fact happens to be true have been provided only very recently.[1,10]

In the case of the master equation corresponding results have been well known for a long time.[19,16,17]

Remark 1. The relative H theorems are valid also for a continuous model of the full Boltzmann equation—the Carleman model.[18]

Remark 2. The H theorems imply (more precisely: are equivalent to) the monotone behavior for *all* convex functionals over P_n.[5] Particularly, this applies to the α entropies and so we get their monotone behavior. Some authors[13] investigate this special class of functionals for B equations.

The aim of our investigation is more general: we characterize those P_n-invariant evolution equations the state solution of which fulfil the relative H theorems. Then, as we shall prove, this fact gives strong implications as to the behavior of the solutions in the large scale (asymptotic behavior, etc.).

Some notions (e.g., stochastic matrix, stochastic generator) are explained in the Appendix.

2. EVOLUTION EQUATIONS IN THE STATE SPACE AND RELATIVE H THEOREMS: A STRUCTURE THEOREM

We suppose that the system of ordinary differential equations

$$(d/dt)\mathbf{p} = \mathbf{v}(\mathbf{p}) \tag{4}$$

is an evolution equation in the state space P_n. For simplicity we suppose further that the vector field $\mathbf{v}: R^n \ni \mathbf{p} \to \mathbf{v}(\mathbf{p}) \in R^n$ is continuously differentiable.

Now we raise the following question: Which structure has the vector field $\mathbf{v}(\mathbf{p})$ for $\mathbf{p} \in P_n$, when all solutions of (4) which start in P_n (and for an evolution equation in the state space also remain there) obey the $\mathbf{q}$-relative H theorems for a strictly positve state $\mathbf{q}$?

Roughly speaking, the following theorem says, that for such P_n-invariant evolution equations (4) fulfilling the $\mathbf{q}$-relative H theorems the vector field $\mathbf{v}(\mathbf{p})$ for $\mathbf{p} \in P_n$ is almost everywhere given by a state-dependent stochastic generator $L(\mathbf{p})$:

$$\mathbf{v}(\mathbf{p}) = L(\mathbf{p})\mathbf{p} \quad \text{and} \quad L(\mathbf{p})\mathbf{q} = \mathbf{0}$$

It is useful to remember (see also the Appendix) that an equation $(d/dt)\mathbf{p} = L\mathbf{p}$ with a (constant) stochastic generator L is a master equation, i.e., a special evolution equation in the state space P_n which fulfils the $\mathbf{q}$-relative H theorems for all state solutions with respect to a stationary state $\mathbf{q}$ (i.e., $L\mathbf{q} = \mathbf{0}$).

Theorem 2.1. Suppose $\mathbf{v}$ is a vector field of an evolution equation in the state space P_n. All solutions of this equation which start in the state space P_n fulfil the $\mathbf{q}$-relative H theorems ($\mathbf{q} > \mathbf{0}$) if and only if there exists an open, dense subset S of P_n and a map $L : S \ni \mathbf{p} \rightarrow L(\mathbf{p})$ from S into the real $n \times n$ matrices with the properties:

(i) $$L(\mathbf{p}) \text{ is a stochastic generator} \tag{5}$$

(ii) $$L(\mathbf{p})\mathbf{q} = \mathbf{0} \tag{6}$$

(iii) $$\mathbf{v}(\mathbf{p}) = L(\mathbf{p})\mathbf{p} \qquad \text{for all} \quad \mathbf{p} \in S$$

For the proof we need some more technical results.

Lemma 2.1. Suppose L is a stochastic generator on R^n with $L\mathbf{q} = \mathbf{0}$ and $\mathbf{q} > \mathbf{0}$. Then, for every convex, differentiable function f on R_+ we have

$$\sum_{j=1}^{n} (L\mathbf{p})_j f'(p_j/q_j) \leqslant 0 \qquad \text{for all} \quad \mathbf{p} \in P_n \tag{7}$$

where f' indicates the derivative of f.

Proof. A differentiable f (at $x = 0$ differentiable from the right) on R_+ is convex iff

$$(t - s)f'(t) \geqslant f(t) - f(s) \qquad \text{for all} \quad t, s \in R_+ \tag{8}$$

We may be content with showing (7) for L with $|L_{ik}| \leqslant 1$ for all i, k (for $\lambda > 0$, λL is a stochastic generator, too). Then there is a stochastic matrix B such that

$$L = -\mathbf{1} + B, \qquad B\mathbf{q} = \mathbf{q} \tag{9}$$

where $\mathbf{1}$ denotes the matrix (δ_{ik}). We define $c_{ik} = q_i^{-1} B_{ik} q_k$ for all i, k, and we get

$$c_{ik} \geqslant 0 \qquad \forall i,k; \qquad \sum_{k=1}^{n} c_{ik} = 1 \qquad \forall i \tag{10}$$

$$\sum_{i=1}^{n} q_i c_{ik} = q_k \qquad \forall k \tag{11}$$

Now our conclusions run along the following line:

$$\begin{aligned}
&\sum_{i,j=1}^{n} L_{ij} p_j f'(p_i/q_i) \\
&\quad = \sum_{i,j=1}^{n} q_i \{ c_{ij}(p_j/q_j) - \delta_{ij}(p_j/q_j) \} f'(p_i/q_i) \\
&\quad = \sum_{i=1}^{n} q_i \left[\sum_{j=1}^{n} c_{ij}(p_j/q_j) - (p_i/q_i) \right] f'(p_i/q_i) \qquad [\text{by (8) and } p_i \geqslant 0] \\
&\quad \leqslant \sum_{i=1}^{n} q_i \left\{ f\left[\sum_{j=1}^{n} c_{ij}(p_j/q_j) \right] - f(p_i/q_i) \right\} \qquad [\text{by convexity and (10)}] \\
&\quad \leqslant \sum_{i=1}^{n} q_i \left[\sum_{j=1}^{n} c_{ij} f(p_j/q_j) - f(p_i/q_i) \right] \qquad [\text{by (11)}] \\
&\quad \leqslant \sum_{j=1}^{n} q_j f(p_j/q_j) - \sum_{i=1}^{n} q_i f(p_i/q_i) = 0 \quad \blacksquare
\end{aligned}$$

Lemma 2.2. Let $\{L^{(m)}\}$ be a sequence of stochastic generators over R^n. Assume $L^{(m)}\mathbf{q} = \mathbf{0}$ for all m ($\mathbf{q} > \mathbf{0}$), and suppose $\mathbf{p} \geqslant \mathbf{0}$ is given such that $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_j = p_j/q_j$ is a nondegenerated vector, i.e., $\lambda_i \neq \lambda_j$ $\forall i \neq j$. Then $\sup_m \|L^{(m)}\mathbf{p}\| < \infty$ implies $\sup_m \|L^{(m)}\| < \infty$, where $\|\cdot\|$ means an appropriate norm in the Euclidean R^n.

Proof. For convenience we may assume $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. We define $n \times n$ matrices $A^{(m)} = (A_{ik}^{(m)})$ by $A_{ik}^{(m)} = L_{ik}^{(m)} q_k$, and we obtain

$$A_{ik}^{(m)} \geqslant 0 \qquad \forall i \neq k, \qquad \sum_{i=1}^{n} A_{ik}^{(m)} = 0 \qquad \forall k, \qquad \sum_{k=1}^{n} A_{ik}^{(m)} = 0 \qquad \forall i \tag{12}$$

$$L^{(m)}\mathbf{p} = A^{(m)}\boldsymbol{\lambda} \qquad \text{for all } m \tag{13}$$

We are going to show that if

$$\sup_m |A_{jj}^{(m)}| < \infty \qquad \text{for} \quad j \geqslant k$$

then

$$\sup_m |A_{jj}^{(m)}| < \infty, \qquad j \geqslant k-1 \tag{14}$$

In fact, in assuming $\sup_m |A_{jj}^{(m)}| < \infty, j \geqslant k$, we obtain from (12)

$$\sup_m |A_{sj}^{(m)}| < \infty, \qquad \sup_m |A_{js}^{(m)}| < \infty, \qquad j \geqslant k, \quad \forall s \tag{15}$$

From (12) we know $A_{k-1\,k-1}^{(m)} = -\sum_{j \neq k-1} A_{k-1\,j}^{(m)}$, so by means of (13) we get

$$(L^{(m)}\mathbf{p})_{k-1} = \sum_{j<k-1} A_{k-1\,j}^{(m)}(\lambda_j - \lambda_{k-1}) + \sum_{j>k-1} A_{k-1\,j}^{(m)}(\lambda_j - \lambda_{k-1}) \tag{16}$$

Owing to (12), (15), and positivity of $\lambda_j - \lambda_{k-1}$ for $j < k-1$, the second part of the right-hand side of (16) is uniformly bounded, whereas the first part is nonnegative. But then, owing to $\sup_m |(L^{(m)}\mathbf{p})_{k-1}| < \infty$ also the first part of the right-hand side of (16) remains bounded. Once more inserting positivity of $\lambda_j - \lambda_{k-1}$ for $j < k-1$ we may conclude that $\sup_m |A_{k-1\,j}^{(m)}| < \infty$ for $j < k-1$. From (12) and $\sup_m |A_{k-1\,j}^{(m)}| < \infty$ for $j > k-1$ it follows that $\sup_m |A_{k-1\,k-1}^{(m)}| < \infty$, so the implication (14) is seen to be true. Now, $(L^{(m)}\mathbf{p})_n = \sum_{j<n} A_{nj}^{(m)}(\lambda_j - \lambda_n)$, and $\lambda_j - \lambda_n > 0$ for $j < n$. This, together with $\sup_m \|L^{(m)}\mathbf{p}\| < \infty$ and (12) implies $\sup_m |A_{nj}^{(m)}| < \infty, j < n$, and once more again making use of (12) we obtain $\sup_m |A_{nn}^{(m)}| < \infty$. In applying implication (14) successively (starting with $k = n$ in (14)) we then arrive at $\sup_m |A_{jj}^{(m)}| < \infty\ \forall j$, from which the result $\sup_m |A_{jk}^{(m)}| < \infty\ \forall j,k$ follows by means of (12). By definition of A this is equivalent with $\sup_m |L_{jk}^{(m)}| < \infty$ $\forall j,k$. ■

Lemma 2.3. Let $\mathbf{p}', \mathbf{p}'', \mathbf{q}$ be probability vectors, $\mathbf{q} > \mathbf{0}$. If

$$S_f(\mathbf{p}'; \mathbf{q}) \leqslant S_f(\mathbf{p}''; \mathbf{q})$$

holds for any continuous, convex f on R_+, there exists a stochastic $n \times n$ matrix T with

$$T\mathbf{p}'' = \mathbf{p}' \quad \text{and} \quad T\mathbf{q} = \mathbf{q}$$

Moreover, if there is a stochastic matrix T such that $T\mathbf{q} = \mathbf{q}$ and $T\mathbf{p}'' = \mathbf{p}'$, then

$$S_f(\mathbf{p}'; \mathbf{q}) \leqslant S_f(\mathbf{p}''; \mathbf{q})$$

is satisfied for *any* convex f on R_+.

Proof. The first part of our assertion is far from being trivial. Therefore, we omit a proof and refer in this respect to Refs. 3, 5, and 15. To see $S_f(T\mathbf{p}''; \mathbf{q}) \leqslant S_f(\mathbf{p}''; \mathbf{q})$ for *any* convex f on R_+, we define $A = (A_{ik})$ by $A_{ik} = q_k T_{ik} q_i^{-1}$. Then, $p_i'/q_i = (T\mathbf{p}'')_i/q_i = \sum_{k=1}^n A_{ik}(p_k''/q_k)$, and if we make use of $A_{ik} \geqslant 0$, $\sum_{k=1}^n A_{ik} = 1$ (this being a consequence of $T\mathbf{q} = \mathbf{q}$), stochasticity of T and convexity of f, the following conclusion is an obvious one:

$$\begin{aligned}S_f(T\mathbf{p}''; \mathbf{q}) &= \sum_{i=1}^{n} q_i f(p_i'/q_i) = \sum_{i=1}^{n} q_i f\left[\sum_{k=1}^{n} A_{ik}(p_k''/q_k)\right] \\ &\leqslant \sum_{i=1}^{n} q_i\left[\sum_{k=1}^{n} A_{ik} f(p_k''/q_k)\right] = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} q_i A_{ik}\right) f(p_k''/q_k) \\ &= \sum_{k=1}^{n} q_k f(p_k''/q_k) = S_f(\mathbf{p}''; \mathbf{q}).\end{aligned}$$

This proves the second part of Lemma 2.3. ■

Let us now come to the proof of our theorem.

Proof of Theorem 2.1. Assume there is some open, dense set $S \subset P_n$, and there exists a map acting from S into $n \times n$ matrices with properties (i)–(iii) from the theorem. Let $0 \leqslant t \to \mathbf{p}(t)$ be a solution of (4) with $\mathbf{p}(0) \in P_n$. Fix $t_0 \geqslant 0$. For every continuous differentiable convex f we have

$$(d/dt)S_f(\mathbf{p}(t); \mathbf{q})|_{t=t_0} = \sum_{i=1}^{n} v_i(\mathbf{p}(t_0)) f'(p_i(t_0)/q_i) \tag{17}$$

Suppose $\{\mathbf{p}^{(m)}\} \subset S$ with $\lim_m \mathbf{p}^{(m)} = \mathbf{p}(t_0)$. Then, as a consequence of (17) we get

$$(d/dt)S_f(\mathbf{p}(t); \mathbf{q})\Big|_{t=t_0} = \lim_{m\to\infty} (d/dt)S_f\big(\mathbf{p}^{(m)}(t); \mathbf{q}\big)\Big|_{t=0} \tag{18}$$

where $0 \leqslant t \to \mathbf{p}^{(m)}(t)$ means the solution of (4) with initial value $\mathbf{p}^{(m)}(0) = \mathbf{p}^{(m)}$. Since $\mathbf{p}^{(m)} \in S$, it follows from property (iii) that $\mathbf{v}(\mathbf{p}^{(m)}) = L(\mathbf{p}^{(m)})\,\mathbf{p}^{(m)}$. Inserting this into (18) and applying Lemma 2.1—the use of which is justified by (5), (6)—we arrive at

$$(d/dt)S_f(\mathbf{p}(t); \mathbf{q})|_{t=t_0} = \lim_{m\to\infty} \sum_{j=1}^{n} \big(L(\mathbf{p}^{(m)})\mathbf{p}^{(m)}\big)_j f'\big(p_j^{(m)}/q_j\big) \leqslant 0$$

The latter has to hold for any $t_0 \geqslant 0$ and for every $\mathbf{p}(0) \in P_n$, with f arbitrary chosen from the continuously differentiable, convex functions. Hence

$$S_f(\mathbf{p}(t); \mathbf{q}) \leqslant S_f(\mathbf{p}(s); \mathbf{q}) \qquad \text{for all} \quad t \geqslant s \geqslant 0 \tag{19}$$

for all $\mathbf{p}(0) \in P_n$, and any continuously differentiable convex f. Since any continuous, convex f can be uniformly approximated by continuously differentiable, convex functions on every finite interval, the inequality (19) persists to hold for continuous, convex functions on R_+. Finally (by Lemma 2.3), we see that (19) extends to all convex functions on R_+. Thus, sufficiency is proven.

Now we prove the other direction.

Let a set S be defined by

$$S = \{\mathbf{x} \in P_n : (x_j/q_j) \neq (x_k/q_k), j \neq k\}$$

This set S is (relatively) open and dense within P_n. Fix $\mathbf{p} \in S$, but arbitrary. We consider the solution $0 \leqslant t \to \mathbf{p}(t)$ of (4) for $\mathbf{p}(0) = \mathbf{p}$. Let $\{t_m\}$ be a strictly decreasing sequence of nonnegative reals, with $\lim_m t_m = 0$. Then, $S_f(\mathbf{p}(t_m); \mathbf{q}) \leqslant S_f(\mathbf{p}; \mathbf{q})$ for all m and for any convex f on R_+. According to Lemma 2.3, we find stochastic matrices $B^{(m)}$ such that

$$\mathbf{p}(t_m) = B^{(m)}\mathbf{p}, \qquad B^{(m)}\mathbf{q} = \mathbf{q}, \qquad \text{for all } m.$$

Hence $t_m^{-1}(\mathbf{p}(t_m) - \mathbf{p}) = t_m^{-1}(B^{(m)} - \mathbf{1})\mathbf{p} = L^{(m)}\mathbf{p}$, where $L^{(m)} := t_m^{-1}(B^{(m)} - \mathbf{1})$ is a stochastic generator with $L^{(m)}\mathbf{q} = 0$. Since $t_m \to 0$, we have

$$\mathbf{v}(\mathbf{p}) = (d/dt)\mathbf{p}|_{t=0} = \lim_{m\to\infty} L^{(m)}\mathbf{p}.$$

This is only possible if $\sup_m \|L^{(m)}\mathbf{p}\| < \infty$.

Since $\mathbf{p} \in S$, Lemma 2.2 becomes applicable, hence $\sup_m \|L^{(m)}\| < \infty$. Therefore, we find a convergent subsequence $(L^{(m_j)})$, $L^{(m_j)} \to L(\mathbf{p})$. Obviously $L(\mathbf{p})\mathbf{q} = \mathbf{0}$, and $L(\mathbf{p})$ is a stochastic generator with $v(\mathbf{p}) = L(\mathbf{p})\mathbf{p}$. Since $\mathbf{p} \in S$ was arbitrarily chosen, necessity is proven. ■

3. RELATIVE H THEOREMS: THE ASYMPTOTIC BEHAVIOR OF TRAJECTORIES

Let $0 \leqslant t \to \mathbf{p}(t) \in P_n$ be a continuous trajectory in the state space (at $t = 0$ continuous from the right). We write briefly $(\mathbf{p}(t))_{t\geqslant 0}$. Now, let $\mathbf{q} \in P_n$ be a fixed, strictly positive vector ($\mathbf{q} > \mathbf{0}$).

Then, we say (Definition 2.1) that the $\mathbf{q}$-relative H theorems are fulfilled for the trajectory $(\mathbf{p}(t))_{t\geqslant 0}$, when $S_g(\mathbf{p}(t); \mathbf{q}) \leqslant S_g(\mathbf{p}(s); \mathbf{q})$ for any convex function g defined on R_+ and for all $t \geqslant s \geqslant 0$.

The main result of this section will be to show that trajectories in the state space satisfying $\mathbf{q}$ relative H theorems behave very regularly for $t \to \infty$.

Theorem 3.1. Let $(\mathbf{p}(t))_{t\geqslant 0}$ be a trajectory in the state space satisfying the $\mathbf{q}$-relative H theorems ($\mathbf{q} > \mathbf{0}$). Then (i) $\mathbf{p}_\infty = \lim_{t\to\infty}\mathbf{p}(t)$ exists; and (ii) $\mathbf{p}(0) > \mathbf{0}$ implies $\mathbf{p}(t) > \mathbf{0}$ for all $t \geqslant 0$ and $\mathbf{p}_\infty > \mathbf{0}$ (strict positivity!).

Proof. We show (i). Let $\Omega(\mathbf{p})$ be the ω-limit set of $(\mathbf{p}(t))_{t\geqslant 0}$, i.e.,

$$\Omega(\mathbf{p}) = \left\{\mathbf{p}' \in R^n : t_1 < t_2 < \cdots, \lim_{m\to\infty} t_m = \infty, \lim_{m\to\infty} \mathbf{p}(t_m) = \mathbf{p}'\right\}$$

Since P_n is compact, and $(\mathbf{p}(t))_{t\geqslant 0} \subset P_n$, we may refer to a basic result from topological dynamics[11] which applies to our situation with the result

$$\Omega(\mathbf{p}) \text{ is a nonvoid, compact, and connected subset of } P_n. \tag{20}$$

Now, we show (with special relative H functionals) that $\Omega(\mathbf{p})$ can only be a one-point set.

Let $\beta \geqslant 0$ be a real. We define a continuous, convex function f by

$$f_\beta(x) = (1/2)(x - \beta + |x - \beta|) \qquad \text{for all} \quad x \in R_+$$

The corresponding S_f functional reads as

$$S_{f_\beta}(\mathbf{p}(t); \mathbf{q}) = \|(\mathbf{p}(t) - \beta q)_+\|_1 \qquad \forall \beta \geqslant 0, \quad \forall t \geqslant 0 \tag{21}$$

where $\mathbf{a}_+$ for $\mathbf{a} \in R^n$ is defined by $\mathbf{a}_+ = \sum_{a_j \geqslant 0} a_j$, and $\|\mathbf{a}\|_1$ means $\sum_{j=1}^n |a_j|$, the L^1 norm of the vector $\mathbf{a}$.

By the assumption of Theorem 3.1, for any $p' \in \Omega(\mathbf{p})$ we have

$$\lim_{t \to \infty} S_{f_\beta}(\mathbf{p}(t); \mathbf{q}) = S_{f_\beta}(\mathbf{p}'; \mathbf{q}) \tag{22}$$

In fact $S_{f_\beta}(\mathbf{p}; \mathbf{q})$ depends continuously on $\mathbf{p}$; hence

$$\lim_{m \to \infty} S_{f_\beta}(\mathbf{p}(t_m); \mathbf{q}) = S_{f_\beta}(\mathbf{p}'; \mathbf{q}), \qquad \text{for} \quad \mathbf{p}' \in \Omega(\mathbf{p}),$$

$$\lim_{m \to \infty} \mathbf{p}(t_m) = \mathbf{p}', \qquad \lim_m t_m = \infty$$

since $S_{f_\beta}(\mathbf{n}(t); \mathbf{q})$ decreases in time, (22) follows.

Taking into account (21), with regard to (22) we may draw the following conclusion:

$$\forall \mathbf{p}', \mathbf{p}'' \in \Omega(\mathbf{p}): \qquad \|(\mathbf{p}' - \beta \mathbf{q})_+\|_1 = \|(\mathbf{p}'' - \beta \mathbf{q})_+\|_1 \qquad \forall \beta \geqslant 0 \tag{23}$$

Let us define $G(\beta) := \|(\mathbf{p}' - \beta \mathbf{q})_+\|_1$, $\mathbf{p}' \in \Omega(\mathbf{p})$. Then, $G(\beta)$ is monotonously decreasing for β increasing. One also easily verifies that $G(\beta)$ is a piecewise linear function, with at most n corners, say, m, at $\beta_1, \ldots, \beta_m > 0$. Suppose $\mathbf{p}', \mathbf{p}'' \in \Omega(\mathbf{p})$. Obviously we may decompose $\{1, \ldots, n\}$ into m mutually disjoint, nonvoid subsets K_j' (K_j'', respectively), $j = 1, \ldots, m$, such that

$$p_i' = \beta_s q_i \qquad \text{for all} \quad i \in K_s', \qquad p_j'' = \beta_s q_j \qquad \text{for all} \quad j \in K_s'' \tag{24}$$

and $s \leqslant m$.

If we now think of $\mathbf{p}'$ as being fixed for the moment, we obtain from (24) that

$$p_j'' = (p_i'/q_i) q_j = (q_j/q_i) p_i' \qquad \text{for any} \quad j \in K_s'', \quad i \in K_s', \quad \forall s \tag{25}$$

has to hold for every $\mathbf{p}''$ and the corresponding decomposition $(K_s'')_{s \leqslant m}$.

Now, there are only finitely many possibilities for decomposing the set $\{1, \ldots, n\}$ into m disjoint, nonvoid subsets. Hence, there is only a finite number of $\mathbf{p}''$ which can obey equations (25). Thus $\Omega(\mathbf{p})$ has finitely many elements. This is consistent with (20) if and only if $\Omega(\mathbf{p})$ is a one-point set: $\Omega(\mathbf{p}) = \{\mathbf{p}_\infty\}$. The relation $\lim_{t\to\infty}\mathbf{p}(t) = \mathbf{p}_\infty$, i.e., (i) follows then since we are working in a compact set P_n.

We show (ii). Assuming $q_1 \leqslant q_j \leqslant q_n$ for all j, and defining $p_i := p_i(0) \geqslant \epsilon > 0$ for all i, we are claiming that $p_i(t) \geqslant (q_1/q_n)\epsilon$ for all i and for all $t > 0$. In order to see (26) we suppose—in contradiction with (26)—that an index i and an instant $t > 0$ exist such that $p_i(t) < (q_1/q_n)\epsilon$. Then, it is easy to construct a nonnegative, convex, and monotonously decreasing, continuous function f on R_+ such that

$$f(\epsilon/q_n) = 1, \qquad f(p_i(t)/q_1) = (2/q_1) \tag{27}$$

For such f we would have

$$\begin{aligned} S_f(\mathbf{p}(t);\mathbf{q}) &= \sum_j q_j f(p_j(t)/q_j) && \text{(positivity of } f) \\ &\geqslant q_i f(p_i(t)/q_i) \geqslant q_i f(p_i(t)/q_1) && [\text{by (27)}] \\ &\geqslant 2(q_i/q_1) \geqslant 2 > 1 = f(\epsilon/q_n) \end{aligned}$$

i.e.,

$$S_f(\mathbf{p}(t);\mathbf{q}) > f(\epsilon/q_n) \tag{28}$$

On the other hand, one may conclude also as follows:

$$\begin{aligned} S_f(\mathbf{p};\mathbf{q}) &= \sum_j q_j f(p_j/q_j) && \text{(monotonicity of } f,\ q_j \leqslant q_n) \\ &\leqslant \sum_j q_j f(p_j/q_n) && \text{(monotonicity of } f,\ p_j \geqslant \epsilon) \\ &\leqslant \sum_j q_j f(\epsilon/q_n) = f(\epsilon/q_n) \end{aligned}$$

i.e.,

$$S_f(\mathbf{p}(0);\mathbf{q}) = S_f(\mathbf{p};\mathbf{q}) \leqslant f(\epsilon/q_n) \tag{29}$$

Now, f is a convex function, and $(\mathbf{p}(t))_{t\geqslant 0}$ satisfies the $\mathbf{q}$-relative H theorems; hence

$$S_f(\mathbf{p}(t);\mathbf{q}) \leqslant S_f(\mathbf{p}(0);\mathbf{q}) \leqslant f(\epsilon/q_n) \qquad [\text{by means of (29)}]$$

This contradicts (28), so our supposition cannot be true. This proves (26). By (i) we then have

$$p_{\infty i} := \lim_{t\to\infty} p_i(t) \geqslant (q_1/q_n)\epsilon$$

i.e., $\mathbf{p}_\infty$ is strictly positive. ■

Up to now (in this section) we have been talking only about trajectories in the state space. Now, we investigate the case that we have an evolution equation in the state space (4) all solutions of which obey the $\mathbf{q}$-relative H theorems. From Theorem 3.1 then follows:

Proposition 3.1. Suppose we have an evolution equation in the state space (4) all solutions of which (starting in the state space) obey the $\mathbf{q}$-relative H theorems ($\mathbf{q} > \mathbf{0}$). Then (i) every solution with $\mathbf{p}(0) \in P_n$ converges towards a stationary state.

(ii) Every solution with strictly positive initial state does not intersect the boundary of P_n at any time (especially: the asymptotic stationary state is strictly positive).

(iii) $\mathbf{q}$ is also a stationary state, and if $\mathbf{q}$ is the *only strictly positive* stationary state of (4), every solution of (4) with strictly positive initial state approaches $\mathbf{q}$ as t goes to infinity.

Proof. From Theorem 3.1 we know that all state solutions converge and do not intersect the boundary of P_n, when they start in the interior of P_n.

We show that the asymptotic states are stationary states. Suppose for a moment $v_i(\mathbf{p}_\infty) > 0$ for an index i. By continuity of $\mathbf{v}$, there is a $\delta > 0$, and $t(\delta) > 0$, such that $v_i(\mathbf{p}(t)) \geqslant \delta$ for all $t \geqslant t(\delta)$, hence $(d/dt)p_i(t) \geqslant \delta > 0$ for all $t \geqslant t(\delta)$ contradicting the fact that $p_i(t)$ has to be bounded with respect to t. Therefore, $v_i(\mathbf{p}_\infty) \leqslant 0$ for all i. If $v_i(\mathbf{p}_\infty) < 0$ for some i, we conclude $p_{\infty i} < 0$ (with similar arguments as above), a contradiction to positivity preservation of (4). Hence, $v(\mathbf{p}_\infty) = \mathbf{0}$ has to hold, i.e., $\mathbf{p}_\infty$ is a stationary state.

Now to (iii). That $\mathbf{q}$ is a stationary state of (4) follows if we look on the behavior of $S_f(\mathbf{q}(t); \mathbf{q})$ with convex $f(s) = |s-1|$, where $0 \leqslant t \to \mathbf{q}(t)$ is the solution of (4) with $\mathbf{q}(0) = \mathbf{q}$. Indeed, for such f, $S_f(\mathbf{q}(t); \mathbf{q}) = \|\mathbf{q}(t) - \mathbf{q}\|_1$ and $S_f(\mathbf{q}(t); \mathbf{q}) \leqslant S_f(\mathbf{q}; \mathbf{q}) = 0$ then implies $\mathbf{q}(t) = \mathbf{q}$ for all $t \geqslant 0$. Hence, $\mathbf{q}_\infty = \mathbf{q}$. The rest of the assertion is evident. ■

Because of stability properties the reference state $\mathbf{q}$ plays an exposed role among all stationary states.

Proposition 3.2. Suppose we have an evolution equation in the state space (4) all solutions of which (starting in P_n) obey the $\mathbf{q}$-relative H theorems ($\mathbf{q} > \mathbf{0}$). Assume there exist only *finitely* many strictly positive stationary state of this equation. Then, $\mathbf{q}$ is asymptotically stable and the only stable stationary state in P_n.

Remark. Of course, we define the notions "stable" and "asymptotically stable" only in restriction to the state space P_n.

Further, it is useful to compare our Proposition 3.2 with the method of Lyapunov functionals in stability theory.

Proof. Take the convex function $f(s) = (s-1)^2$. Then, $S_f(\mathbf{p};\mathbf{q}) = \sum_i q_i^{-1}(p_i - q_i)^2$. Define M as the set of stationary states of (4) (in P_n) which are different from $\mathbf{q}$. Since $\mathbf{q}$ is not a limit point of M, we have $\epsilon = \inf_{\mathbf{q}' \in M} S_f(\mathbf{q}';\mathbf{q}) > 0$. Let U be given as $U = \{\mathbf{p} \in P_n : S_f(\mathbf{p};\mathbf{q}) < \epsilon\}$. U is a neighborhood of $\mathbf{q}$ in P_n. Assume $\mathbf{p} \in U$, and $\lim_{t\to\infty}\mathbf{p}(t) = \mathbf{p}_\infty$ (cf. Proposition 3.1)—where $\mathbf{p}(t)$ is the solution of (4) for $\mathbf{p}(0) = \mathbf{p}$. Then, $\mathbf{p}_\infty$ is a stationary state, and

$$S_f(\mathbf{p}_\infty;\mathbf{q}) \leqslant S_f(\mathbf{p}(0);\mathbf{q}) < \epsilon, \qquad \text{i.e.,} \quad \mathbf{p}_\infty \notin M$$

Hence $\mathbf{p}_\infty = \mathbf{q}$.

If $\|\cdot\|_2$ denotes the L^2 norm in R^n, we see

$$\left(\max_j q_j\right)^{-1}\|\mathbf{p}-\mathbf{q}\|_2^2 \leqslant S_f(\mathbf{p};\mathbf{q}) \leqslant \left(\min q_j\right)^{-1}\|\mathbf{p}-\mathbf{q}\|_2^2$$

Therefore, $S_f(\mathbf{p}(t);\mathbf{q}) \leqslant S_f(\mathbf{p}(0);\mathbf{q})$ for all $t \geqslant 0$ implies $\|\mathbf{p}(t)-\mathbf{q}\|_2 < \delta$ for all $t \geqslant 0$ whenever

$$\|\mathbf{p}(0)-\mathbf{q}\|_2 \leqslant \delta\left(\min_i q_i \Big/ \max_j q_j\right)^{1/2}$$

Thus, $\mathbf{q}$ is an asymptotic stable state.

Let $\mathbf{q}' \in M$, and suppose $0 < \vartheta < 1$, $\mathbf{p}^\vartheta = (1-\vartheta)\mathbf{q}' + \vartheta\mathbf{q}$. For sufficiently small ϑ we have

$$\mathbf{p}^\vartheta \notin (\text{boundary of } P_n), \quad \mathbf{p}^\vartheta \notin M, \quad \text{and} \quad \|\mathbf{p}^\vartheta - \mathbf{q}'\|_2 = \vartheta\|\mathbf{q}-\mathbf{q}'\|_2 \tag{30}$$

Since $S_f(\boldsymbol{\lambda};\mathbf{q})$ is convex in $\boldsymbol{\lambda} \in P_n$, we have

$$\begin{aligned} S_f(\mathbf{p}^\vartheta;\mathbf{q}) &\leqslant (1-\vartheta)S_f(\mathbf{q}';\mathbf{q}) + \vartheta S_f(\mathbf{q};\mathbf{q}) = (1-\vartheta)S_f(\mathbf{q}';\mathbf{q}) + 0 \\ &< S_f(\mathbf{q}';\mathbf{q}) \neq 0 \end{aligned}$$

If $0 \leqslant t \to \mathbf{p}^\vartheta(t)$ is the solution of (4) for $\mathbf{p}^\vartheta(0) = \mathbf{p}^\vartheta$, the just-derived estimate and Proposition 3.1 result in the following: If $\lim_{t\to\infty}\mathbf{p}^\vartheta(t) = \mathbf{p}^\vartheta_\infty$, then $S_f(\mathbf{p}^\vartheta_\infty;\mathbf{q}) < S_f(\mathbf{q}';\mathbf{q})$. Hence, $\mathbf{p}^\vartheta \neq \mathbf{q}'$. Owing to Proposition 3.1 $\mathbf{p}^\vartheta \notin$ (boundary of P_n) implies $\mathbf{p}^\vartheta_\infty \notin$ (boundary of P_n), and we may conclude that ($M' := M \cup \{\mathbf{q}\}$)

$$\lim_{t\to\infty}\|\mathbf{p}^\vartheta(t) - \mathbf{q}'\|_2 \geqslant \inf_{\substack{\mathbf{q}^1 \neq \mathbf{q}^2 \\ \mathbf{q}^1 \in M' \setminus (\text{boundary } P_n) \\ \mathbf{q}^2 \in M'}} \|\mathbf{q}^1 - \mathbf{q}^2\|_2 = m > 0$$

holds for any $\vartheta \in (0,1)$ which is sufficiently small. At the same time, $\lim_{\vartheta\to 0}\|\mathbf{p}^\vartheta - \mathbf{q}'\|_2 = 0$ by (30). Thus $\mathbf{q}'$ cannot be stable, i.e., $\mathbf{q}$ is the only stable state of (4) in P_n. ■

4. A CLASS OF VECTOR FIELDS

Up to now, we always supposed that we had an evolution equation in the state space P_n. Then we obtained further results about asymptotic behavior, etc., when the relative H theorems are fulfilled.

Now, we specify a class of vector fields. It will be shown that these vector fields—which we call vector fields of class (E)—automatically give evolution equations in the state space.

Definition 4.1 [vector fields of class (E)]. $\mathbf{v}$ is a vector fields of class (E) if $\mathbf{v}(\mathbf{p}) = L(\mathbf{p})\mathbf{p}$ for $\mathbf{p} \in R^n$ and $L: \mathbf{p} \to L(\mathbf{p})$ is a continuously differentiable map from R^n into the real $n \times n$ matrices fulfilling (i) $\sum_i L(\mathbf{p})_{ik} = 0$ for all k, $\mathbf{p} \in R^n$(!); and (ii) $L(\mathbf{p})$ is a stochastic generator whenever $\mathbf{p} \in P_n$.

Remark. If $\mathbf{p} \to L'(\mathbf{p})$ is a continuously differentiable map from P_n into the $n \times n$-stochastic generators one can always extend this map L' to $L: \mathbf{p} \to L(\mathbf{p})$ over R^n with property (i) (with a reflection method).

Later we prove the following:

Proposition 4.1. Every vector field of class (E) gives an evolution equation in the state space.

With help of Proposition 4.1, the structure Theorem 2.1 (take $S = P_n$) and the Propositions 3.1 and 3.2 we get the following.

Theorem 4.1. Let $\mathbf{v}$ be a vector fields of class (E) which fulfils the additional condition $L(\mathbf{p})\mathbf{q} = 0$ for $\mathbf{q} > 0$ and $\mathbf{p} \in P_n$.

Then, we obtain an evolution equation in the state space. Every solution which starts from a state $\mathbf{p} \in P_n$ is a trajectory that satisfies the $\mathbf{q}$-relative H theorems. Every such solution converges in P_n toward a stationary state. A solution, arising from an inner point, does not intersect the boundary of P_n at any time and approaches an inner stationary point of the equation.

$\mathbf{q}$ itself is a stationary state, and if $\mathbf{v}(\mathbf{x}) = 0$ has only finitely many, strictly positive solutions in P_n, $\mathbf{q}$ is asymptotically stable and the only stable stationary state in P_n.

Now we come to the proof of Proposition 4.1. Firstly, we formulate the following

Lemma 4.1. Let $t \to L(t)$ be a continuous map from R_+ into the $n \times n$-stochastic generators. For $t, s \in R_+$, $t \geqslant s$, let a matrix $T(t,s)$ be defined as

$$T(t,s) = 1 + \sum_{m=1}^{\infty} \int_s^t L(s_1) \int_s^{s_1} L(s_2) \cdots \int_s^{s_{m-1}} L(s_m)\, ds_1 \ldots ds_m$$

where $s_0 = t$.

Then, $T(t,s)$ is a stochastic $n \times n$ matrix.

We shall not give a detailed proof of this useful fact, merely indicate the essential steps of the argumentations showing the validity of the assertion.

Firstly, one shows validity of an "inhomogeneous" Lie formula

$$T(t,s) = \lim_{N\to\infty} (1 + ((t-s)/N)L(n_N)) \cdots (\mathbf{1} + ((t-s)/N)L(n_1)) \times (\mathbf{1} + ((t-s)/N)L(s)) \qquad (*)$$

where $n_k := s + k((t - s/N)$, $k = 1, \ldots, N$.

We remark that such type of formula holds true also in much more general situations [e.g., for strongly continuous families $(L(t))$ of bounded linear operators on Banach spaces]. Secondly, one notes that $L(x)$ is uniformly continuous on a finite interval $[s,t]$—then, there is N_0 such that $(1 + ((t-s)/N)L(x))$ for all $x \in [s,t]$ is a *stochastic matrix* whenever $N \geqslant N_0$. From this and $(*)$ then our assertion follows.

We are now showing the following:

(I) Let $U \subset R^n$ be an open set, with $P_n \subset U$, and assume $\mathbf{p} \to L(\mathbf{p})$ yields a stochastic generator for all $\mathbf{p} \in U$. Then, the assertion of Proposition 4.1 is true.

Proof of (I). Owing to the usual extension principle for solutions in a compact set (applied in forward direction to the boundary of the compact set) it is sufficient to show that $\mathbf{p}(0) \in$ boundary of P_n implies $\mathbf{p}(t) \in P_n$ for all t of a certain interval $[0,\tau]$, $\tau > 0$. In line with this, let $\mathbf{p}(0) \in$ boundary of P_n, and $0 \leqslant t \to \mathbf{p}(t)$ a local solution of $(d/dt)\mathbf{p} = L(\mathbf{p})\mathbf{p}$ to initial data $\mathbf{p}(0)$. There is $\tau > 0$ with $\mathbf{p}(t) \in U$ for all $t \in [0,\tau]$. Then, $\{L(\mathbf{p}(t)\}_{t\in[0,\tau]}$ is a continuous family of stochastic generators, and by Lemma 4.1

$$T(t,0) = id + \sum_{m=1}^{\infty} \int_0^t L(\mathbf{p}(s_1)) \int_0^{s_1} L(\mathbf{p}(s_2)) \cdots \int_0^{s_{m-1}} L(\mathbf{p}(s_m))\, ds_1 \cdots ds_m$$

is a stochastic matrix for all $t \in [0,\tau]$. Since $(d/dt)\mathbf{p}(t) = L(\mathbf{p}(t))\mathbf{p}(t)$ for $t \leqslant \tau$, necessarily $\mathbf{p}(t) = T(t,0)\mathbf{p}(0)$, so $\mathbf{p}(t) \in P_n$ within $[0,\tau]$.

Proof of Proposition 4.1. Define for $\lambda \in R^1$ a perturbation of our stochastic generator $L(\mathbf{p})$: $L(\mathbf{p},\lambda) = L(\mathbf{p}) + \lambda^2 M$, where M is a stochastic generator with $M_{ik} \neq 0\ \forall i,k$. Then, for fixed $\lambda \neq 0$, $L(\mathbf{p},\lambda)$ is in the set of stochastic generators whenever $\mathbf{p}$ varies through the open set

$$U(\lambda) = \left\{\mathbf{p} \in R^n : -L(\mathbf{p})_{ik} < \lambda^2 M_{ik},\, i \neq k;\, L(\mathbf{p})_{ii} < -\lambda^2 M_{ii}\ \forall i\right\}$$

Further, we have $P_n \subset U$. Let $\lambda \neq 0$. By (I) there exists a uniquely determined global solution $0 \leqslant t \to \mathbf{p}(t,\lambda)$ of $(d/dt)\mathbf{p} = L(\mathbf{p},\lambda)\mathbf{p}$ to initial data $\mathbf{p}(0) \in P_n$, with $\mathbf{p}(t,\lambda) \in P_n$ for all $t \in [0,\infty)$. Owing to $\lim_{\lambda\to 0} L(\mathbf{p},\lambda) = L(\mathbf{p})$ by standard continuity argument we have for the local solution $0 \leqslant t \to \mathbf{p}(t)$ of $(d/dt)\mathbf{p} = L(\mathbf{p})\mathbf{p}$ to initial state $\mathbf{p}(0) \in P_n$: $\mathbf{p}(t) = \lim_{\lambda\to 0}\mathbf{p}(t, \lambda)$, so $\mathbf{p}(t) \in P_n$ in a certain interval. The standard extension principle for solutions then completes the proof. ■

5. QUADRATIC SYSTEMS

In this part we want to illustrate all we have derived up to now by some special systems.

We restrict our considerations to such vector fields $\mathbf{v}$ of class (E) which depend quadratically on the variables, at least so far $\mathbf{v}$ is considered in restriction to P_n. This means that $\mathbf{p} \to L(\mathbf{p})$ should be an affine map on P_n. Therefore, $L(\mathbf{p}) = L(\sum_k p_k \mathbf{e}_k) = \sum_k p_k L(\mathbf{e}_k) = \sum_k p_k L_k$, with $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_n = (0, \ldots, 0, 1)$ and $\{L_k\}$—a set of stochastic generators.

From Proposition 4.1 we know that such quadratic equations are evolution equations in the state space. Suppose that $L_k\mathbf{q} = \mathbf{0}\ \forall k$, $\mathbf{q} > \mathbf{0}$—we can formulate the equivalent of Theorem 4.1.

Theorem 5.1. Let $L_1, \ldots, L_n$ be stochastic generators with $L_k\mathbf{q} = \mathbf{0}$ for all k, $\mathbf{q}$ some fixed, strictly positive state.

(i) Then, $(d/dt)\mathbf{p} = \sum_{k=1}^{n} p_k L_k \mathbf{p}$ is an evolution equation in the state space. Every solution which evolves in P_n satisfies the $\mathbf{q}$-relative H theorems and converges as t tends to infinity $(+\infty)$.

(ii) If $(\sum_k L_k)_{ij} \neq 0$ for all i, j, every solution which has strictly positive initial condition tends towards $\mathbf{q}$.

Proof. We have to show only part (ii). When we take into account that $L_{k,ij} \geqslant 0\ \forall i \neq j$, $L_{k,ii} \leqslant 0\ \forall i$ and in both cases for all k—the assertion follows from the following fact:

Assume for any j,k there exists an index l such that $L_{l,jk} \neq 0$. Then $\mathbf{v}(\mathbf{p}) = 0$ has exactly one strictly positive solution in P_n: $\mathbf{p} = \mathbf{q}$.

We prove this assertion.

Take $0 < \lambda < 1$ such that $\lambda < (1/\max_{l,j,k}|L_{l,jk}|)$. Then, $B(\mathbf{p}) := \mathbf{1} + \lambda\sum_k p_k L_k$ is a stochastic matrix for $\mathbf{p} \in P_n$, and

$$\mathbf{p} > \mathbf{0} \Rightarrow B(\mathbf{p})_{jk} > 0 \qquad \text{for all } j, k \tag{31}$$

We may apply a standard result from the theory of Markov chains (see, e.g., Refs. 9 and 12), saying that (31) implies $B(\mathbf{p})$ to be ergodic for every $\mathbf{p} \in P_n$, strictly positive, i.e., $\lim_{m\to\infty} B(\mathbf{p})^m \mathbf{p}'' = \lim_{m\to\infty} B(\mathbf{p})^m \mathbf{p}'$ for all $\mathbf{p}', \mathbf{p}'' \in P_n$.

Since $L_k \mathbf{q} = \mathbf{0}$, we have $B(\mathbf{p})\mathbf{q} = \mathbf{q}$. Hence, taking $\mathbf{p}'' = \mathbf{q}$ we obtain

$$\lim_m B(\mathbf{p})^m \mathbf{p}' = \mathbf{q} \qquad \text{for all} \quad \mathbf{p}' \in P_n \tag{32}$$

and for all $\mathbf{p} > \mathbf{0}$.

Assume $\mathbf{v}(\mathbf{p}) = \mathbf{0}$, $\mathbf{p} > \mathbf{0}$. Then, $L(\mathbf{p})\mathbf{p} = \mathbf{0}$, i.e., $B(\mathbf{p})\mathbf{p} = \mathbf{p}$. Thus, by (32) and due to $\mathbf{p} > \mathbf{0}$, $\mathbf{p} = B(\mathbf{p})\mathbf{p} = \lim_m B(\mathbf{p})^m \mathbf{p} = \mathbf{q}$. ■

Example (B equations). Now, we can handle, e.g., the B equation (1) from Section 1. Assume (A_{ijkl}) satisfies (2), (3).

Suppose $B^{(k)} = (B_{il}^{(k)})$ is defined by $B_{il}^{(k)} := \sum_j A_{ijkl}$, and there is $\mathbf{q} > \mathbf{0}$, $\mathbf{q} \in P_n$ such that

$$B^{(k)} \mathbf{q} = \mathbf{q} \qquad \text{for all } k \tag{33}$$

We will refer to this condition as "*mixing*" with respect to $\mathbf{q}$.

Proposition 5.1. A B equation with (2), (3) fulfilling the mixing condition (33) with respect to a strictly positive state $\mathbf{q}$ is an evolution equation in the state space. Every solution which starts with strictly positive initial state fulfils the $\mathbf{q}$-relative H theorems and tends to $\mathbf{q}$ for $t \to \infty$.

Proof. Because of (2), (3), $B^{(k)}$ is stochastic for every k. If we define $L(\mathbf{p}) := \sum_k p_k B^{(k)} - \mathbf{1} = \sum_k p_k (B^{(k)} - \mathbf{1}) = \sum_k p_k L_k$ $(\mathbf{p} \in P_n)$, we see that L_k is a stochastic generator obeying $L_k \mathbf{q} = 0$ for all k due to $B^{(k)} \mathbf{q} = \mathbf{q}$ for all k.

Then, (1) reads as $(d/dt)\mathbf{p} = \sum_k p_k L_k \mathbf{p}$. We show $(\sum_k L_k)_{il} \neq 0$ for all i, l.

In contradiction with this, we suppose that (i, l) exist, with $\sum_k L_{k,il} = 0$. Then, $L_{k,il} = 0$ for all k (L_k's are stochastic generators!). Hence, $B_{il}^{(k)} = \delta_{il} + L_{k,il} = \delta_{il}$ for all k. Because of (2) we have $B_{il}^{(k)} = B_{ik}^{(l)}$, so $B_{ik}^{(l)} = \delta_{il}$ for all k. If $i \neq l$, $B_{ik}^{(l)} = 0$ for all k, i.e., $(B^{(l)}\mathbf{q})_i = 0$, which contradicts (33). If $i = l$, $L_{k,ll} = 0$ for all k, so $\sum_{j \neq l} L_{k,jl} = -L_{k,ll} = 0$, i.e., since $L_{k,jl} \geqslant 0$, $j \neq l$, $L_{k,jl} = 0$ $\forall j$, $\forall k$. Therefore, $B_{jl}^{(k)} = \delta_{jl} + L_{k,jl} = \delta_{jl}$ for all k. By symmetry (2), $B_{jk}^{(l)} = \delta_{jl}$ for all k. If $j = l$, $B_{jk}^{(l)} = 0$ for all k, which leads to the mentioned violation of (33) again. Therefore $(\sum_k L_k)_{il} \neq 0$ for all i, l. All these facts show that Theorem 5.1 is applicable, and the assertion follows. ■

Remark. The transition probability per unit time that the system goes from state l to state i at time t is given by $p(i; l)(t) = \sum_k B_{ik}^{(l)} p_k(t)$. Then, we would say that the process is mixing, when $c_i = \lim_{t\to\infty} p(i; l)(t)$

exists for all i, almost all initial states, and is independent of l. Suppose the "mixing" condition (33) holds for the state $\mathbf{q} > \mathbf{0}$. We already know that in this case $\lim_{t\to\infty}\mathbf{p}(t) = \mathbf{q}$ for $\mathbf{p}(0) > \mathbf{0}$. Then follows

$$c_i = \lim_{t\to\infty} p(i;l)(t) = \sum_k B_{ik}^{(l)} q_k = q_i$$

(for all initial states from the interior of P_n). So, the mixing behavior of the process follows from the condition "mixing." Of course, (33) is a restricting condition—but for every strictly positive state $\mathbf{q}$ there exist B equations fulfilling the $\mathbf{q}$-relative H theorems[1] and suggesting applications in reaction kinetics, etc.

6. CONCLUDING REMARKS

The mixing condition for B equations or the condition $L(\mathbf{p})\mathbf{q} = 0$ $\forall \mathbf{p} \in R^n$ for a vector field of class (E) guarantees that all state solutions (starting in the interior of P_n) fulfil the $\mathbf{q}$-relative H theorems.

A first step into a more general situation without these additional conditions—a situation which contains some of the known discrete velocity gases (e.g., Broadwell model[8])—is the following:

We have a vector field, all solutions of which are simultaneously solutions of (*in general different*) master equations.

It is known that a trajectory in the state space, which is a solution of a master equation, is uniquely characterized by a hierarchy of generalized H theorems—a generalization of our notions in Section 1.

Definition (generalized H theorems). We say that a trajectory $(\mathbf{p}(t))_{t\geqslant 0}$ in P_n *satisfies the generalized H theorems* if and only if for arbitrary natural m, for arbitrary instants of time $t_1, \ldots, t_m$, and all $t \geqslant 0$,

$$\begin{aligned} S_f^{(m)}(\mathbf{p}(t_1+t), \ldots, \mathbf{p}(t_m+t)) &:= \sum_i f(p_i(t_1+t), \ldots, p_i(t_m+t)) \\ &\leqslant S_f^{(m)}(\mathbf{p}(t_1), \ldots, \mathbf{p}(t_m)) \\ &:= \sum_i f(p_i(t_1), \ldots, p_i(t_m)) \end{aligned}$$

for any function f in m variables (defined on R_+^m), which is simultaneously convex in the variables and homogeneous of degree 1.

This fact helps us to answer the following question: What is the structure of the vector fields $\mathbf{v}$ the solutions of which fulfill these generalized H theorems, i.e., the solutions of which are simultaneously solutions of (in general different!) master equations?

The result is[2] as follows:

There exists a map $\mathbf{p} \to L(\mathbf{p})$ from P_n into the $n \times n$-stochastic generators with $\mathbf{v}(\mathbf{p}) = L(\mathbf{p})\mathbf{p}$, $\mathbf{p} \in P_n$ and $L(\mathbf{p})$ is a constant of motion, i.e., $L_{ij}(\mathbf{p}(0)) = L_{ij}(\mathbf{p}(t))$ $\forall i, j$ and for all $t \geqslant 0$, when $\mathbf{p}(t)$ is a solution with $\mathbf{p}(0) \in P_n$.

Remark. It is possible to formulate a certain reverse result (see Ref. 2).

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APPENDIX A. SOME DEFINITIONS

Definition A.1. A real $n \times n$ matrix $A = (A_{ij})$ will be called *stochastic* iff (i) $A_{ij} \geqslant 0$ for all i, j, (ii) $\sum_i A_{ij} = 1$ for all j.

Definition A.2. A real $n \times n$ matrix $L = (L_{ij})$ will be called *stochastic generator* iff (i) $L_{ij} \geqslant 0$, $\forall i \neq j$, (ii) $L_{ii} \leqslant 0$ $\forall i$, (iii) $\sum_i L_{ij} = 0$ $\forall j$.

We remark that a stochastic generator is the generator of a semigroup of stochastic matrices.

Definition A.3. A system of ordinary differential equations in R^n will be called *master equation*, when it has the following structure:

$$(d/dt)\mathbf{p} = L\mathbf{p}$$

where L is a stochastic generator. With a stochastic matrix $A = (A_{ij})$ a master equation can be written (possibly only after absorbing a positive muliplicative constant in t):

$$(d/dt)\mathbf{p} = A\mathbf{p} - \mathbf{p}$$

This notion generalizes somewhat Pauli's master equation.[2]

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